MATH 113 - WORKSHEET 3
SOLUTIONS

(1) Let G be a group, and let $a \in G$. Let λ_a be the function $G \to G$ defined by $\lambda_a(g) = ag$.
  (a) Show that λ_a is a bijection $G \to G$.
  (b) Show that λ_a is *not* a homomorphism in general.
  (c) Is λ_a ever a homomorphism?

  λ_a is called "translation by a", or simply "left multiplication by a".

  Solution:
  (a) To show λ_a is a bijection, it suffices to find an inverse function. We'll show that $\lambda_{a^{-1}} = (\lambda_a)^{-1}$. Let $g \in G$. Then $\lambda_{a^{-1}}(\lambda_a(g)) = a^{-1}ag = g$, and $\lambda_a(\lambda_{a^{-1}}(g)) = aa^{-1}g = g$.
  (b) Note that $\lambda_a(e) = ae = a$. Since homomorphisms preserve the identity, λ_a is not a homomorphism whenever $a \neq e$.
  (c) On the other hand, if $a = e$, then $\lambda_a(g) = eg = g$, so λ_a is the identity homomorphism.

(2) For this problem, use the notation and result of problem 1(a). Show that there is a one-to-one homomorphism $\phi : G \hookrightarrow S_G$, where S_G is the symmetric group on the *set* G, given by $\phi(a) = \lambda_a$.

  This is called *Cayley's theorem*, and it shows that every group is isomorphic to a group of permutations (i.e. a subgroup of a permutation group).

  Solution: By problem 1(a), $\phi(a) = \lambda_a \in S_G$, so ϕ is well-defined.
  ϕ is a homomorphism: Let $a, b \in G$. Then $\phi(ab) = \lambda_{ab}$ and $\phi(a) \circ \phi(b) = \lambda_a \circ \lambda_b$. So we'd like to show that $\lambda_{ab} = \lambda_a \circ \lambda_b$. Indeed, for all $g \in G$, $\lambda_{ab}(g) = abg = \lambda_a(\lambda_b(g))$.
  ϕ is one-to-one: Let $a, b \in G$ with $\phi(a) = \phi(b)$. Then $\lambda_a = \lambda_b$, so in particular $a = \lambda_a(e) = \lambda_b(e) = b$.

(3) Let G be a group, and let $a \in G$. Let τ_a be the function $G \to G$ given by $\tau_a(g) = aga^{-1}$. Show that τ_a is an automorphism of G.

  τ_a is called "conjugation by a".

  Solution: τ_a is a homomorphism: Let $g, g' \in G$. Then $\tau_a(g)\tau_a(g') = (aga^{-1})(ag'a^{-1}) = agg'a^{-1} = \tau_a(gg')$.
  τ_a has an inverse: Indeed, $\tau_{a^{-1}} = (\tau_a)^{-1}$. Let $g \in G$. Then $\tau_a(\tau_{a^{-1}}(g)) = a(a^{-1}g(a^{-1})^{-1})a^{-1} = aa^{-1}gaa^{-1} = g$, and $\tau_{a^{-1}}(\tau_a(g)) = a^{-1}(aga^{-1})(a^{-1})^{-1} = g$.

(4) For this problem, use the notation and result of problem 3.
  (a) Show that there is a homomorphism $\phi : G \to \mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ is the group of automorphisms of G, given by $\phi(a) = \tau_a$.

(b) Show that this homomorphism is *not* one-to-one in general.

An automorphism of the form τ_a is called an "inner automorphism", and $\text{range}(\phi) = \{\tau_a \mid a \in G\} \leq \text{Aut}(G)$ is called the "group of inner automorphisms of G".

Solution:

(a) By problem 3, $\phi(a) = \tau_a \in \text{Aut}(G)$, so ϕ is well-defined.
Let $a, b \in G$. Then $\phi(ab) = \tau_{ab}$, and $\phi(a)\phi(b) = \tau_a \circ \tau_b$. So we'd like to show that $\tau_{ab} = \tau_a \circ \tau_b$. Indeed, for all $g \in G$, $\tau_{ab}(g) = (ab)g(ab)^{-1} = abgb^{-1}a^{-1} = \tau_a(\tau_b(g))$.

(b) For a counterexample, let G be any nontrivial abelian group. Then for any $a \in G$, $\phi(a) = \tau_a = \text{id}_G$, since for $g \in G$, $\tau_a(g) = aga^{-1} = aa^{-1}g = g$. So ϕ is the trivial homomorphism sending all elements of G to the identity of $\text{Aut}(G)$.

(5) In lecture, we showed that A_3 is abelian and cyclic (isomorphic to $\mathbb{Z}_3$). Is A_4 abelian?

Solution: No. $(123) = (13)(12)$ and $(124) = (14)(12)$ are elements of A_4 which do not commute: $(123)(124) = (13)(24)$ and $(124)(123) = (14)(23)$.

A_4 is a nonabelian group of order $4!/2 = 24/2 = 12$ which we have not met before.